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THE INFLUENCE OF WARPING AND WINKLER-PASTERNAK SOIL ON THE TORSIONAL VIBRATIONS OF THIN-WALLED OPEN SECTION BEAMS WITH GUIDED-END CONDITIONS

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ABSTRACT

This paper deals with the free torsional vibrations of doubly symmetric thin-walled beams of open section and resting on Winkler-Pasternak continuous foundation. A general dynamic stiffness matrix is developed in this paper which includes the effects of warping and Winkler-Pasternak foundation on the torsional natural frequencies. The resulting highly transcendental frequency equations for a simply supported and guided-end conditions are solved for varying values of warping Winkler and Pasternak foundation parameters on its frequencies of vibration. A new MATLAB code was developed based on modified BISECTION method to solve the highly transcendental frequency equations and to accurately determine the torsional natural frequencies for various boundary conditions. Numerical results for natural frequencies for various values of warping and Winkler and Pasternak foundation parameters are obtained and presented in graphical form showing their parametric influence clearly.

KEYWORDS: Dynamic Stiffness Matrix, Winkler-Pasternak Foundation, Warping, MATLAB, Bisection Method

INTRODUCTION

The problem of a beam on an elastic foundation is important in both the civil and mechanical engineering fields, since it constitutes a practical idealization for many problems (e.g. the footing foundation supporting group of columns (as shown in **Figure 1**) etc. The concept of beams and slabs on elastic foundations has been extensively used by geotechnical, pavement and railroad engineers for foundation design and analysis.

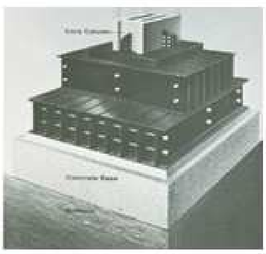


Figure 1: Grillage Foundation

The analysis of structures resting on elastic foundations is usually based on a relatively simple model of the foundation's response to applied loads. A simple representation of elastic foundation was introduced by Winkler(as shown in Figure 2) in 1867 [1]. The Winkler model (one parameter model), which has been originally developed for the analysis of railroad tracks, is very simple but does not accurately represents the characteristics of many practical foundations.

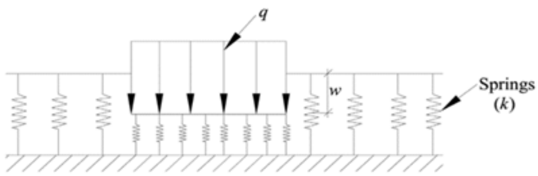


Figure 2: Deflections of Winkler Foundation under Uniform Pressure Q

In order to eliminate the deficiency of Winkler model, improved theories have been introduced on refinement of Winkler's model, by visualizing various types of interconnections such as shear layers and beams along the Winkler springs [1] (Filonenko-Borodich (1940) [2]; Hetényi (1946) [3]; Pasternak (1954) [4]; Kerr (1964) [5]). These theories have been attempted to find an applicable and simple model of representation of foundation medium. To overcome the Winkler model shortcomings improved versions [6] [7] have been developed. A shear layer is introduced in the Winkler foundation and the spring constants above and below this layer is assumed to be different as per this formulation. The following figure shows the physical representation of the Winkler-Pasternak model.

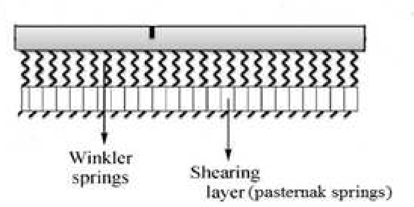


Figure 3: Winkler-Pasternak Model

The vibrations of continuously-supported finite and infinite beams on elastic foundation has wide applications in the design of aircraft structures, base frames for rotating machinery, railroad tracks, etc. Quite a good amount of literature exists on this topic, and valuable practical methods for the analysis of beams on elastic foundation have been suggested. [8-11]. Kameswara Rao et al [12-14] studied the problem of torsional vibration of long, thin-walled beams of open sections resting on Winkler-type elastic foundations using exact, finite element and approximate expressions for torsional frequency of a thin-walled beams and subjected to a time-invariant axial compressive force.

It is well known that a dynamic stiffness matrix is mostly formed by frequency-dependent shape functions which are exact solutions of the governing differential equations. It overcomes the discretization errors and is capable of predicting an infinite number of natural modes by means of a finite number of degrees of freedom. This method has been applied successfully for many dynamic problems including natural vibration. A general dynamic-stiffness matrix of a Timoshenko beam for transverse vibrations was derived including the effects of rotary inertia of the mass, shear distortion, structural damping, axial force, elastic spring and dashpot foundation [15]. Analytical expressions were derived for the coupled bending-torsional dynamic stiffness matrix terms of an axially loaded uniform Timoshenko beam element [16-20] and also a dynamic stiffness matrix is derived based on Bernoulli–Euler beam theory for determining natural frequencies and mode shapes of the coupled bending-torsion vibration of axially loaded thin-walled beams with mono-symmetrical cross sections, by using a general solution of the governing differential equations of motion including the effect of warping stiffness and axial force [21] and [22]. Using the technical computing program Mathematica, a new dynamic stiffness matrix was derived based on the power series method for the spatially coupled free vibration analysis of thin-walled curved beam with non-symmetric cross-section on Winkler and also Pasternak types of elastic foundation [23] and [24]. The free vibration frequencies of a beam were also derived with flexible ends resting on Pasternak soil, in the presence of a concentrated mass at an arbitrary intermediate abscissa [25]. The static and dynamic behaviors of tapered beams were studied using the differential quadrature method (DQM) [26] and also a finite element procedure was developed for analyzing the flexural vibrations of a uniform Timoshenko beam-column on a two-parameter elastic foundation [27].

Though many interesting studies are reported in the literature [8-27], the case of doubly-symmetric thin-walled open section beams resting on Winkler–Pasternak foundation is not dealt sufficiently in the available literature to the best of the author's knowledge.

In view of the above, the present paper deals with the free torsional vibrations of doubly symmetric thin-walled beams of open section and resting on Winkler-Pasternak continuous foundation. A general dynamic stiffness matrix is developed in this paper which includes the effects of warping and Winkler-Pasternak foundation on the torsional natural frequencies. The resulting highly transcendental frequency equations for a simply supported and guided-end conditions are solved for varying values of warping Winkler and Pasternak foundation parameters on its frequencies of vibration. A new MATLAB code was developed based on modified BISECTION method to solve the highly transcendental frequency equations and to accurately determine the torsional natural frequencies for various boundary conditions. Numerical results for natural frequencies for various values of warping and Winkler and Pasternak foundation parameters are obtained and presented in graphical form showing their parametric influence clearly.

FORMULATION AND ANALYSIS

Consider a long doubly-symmetric thin-walled beam of open-section of length L and resting on a Winkler-Pasternak type continuous foundation of Pasternak layer stiffness(K_p), Winkler torsional stiffness (K_w) and undergoing free torsional vibrations. The corresponding differential equation of motion can be written as:

$$EC_w \frac{d^4\emptyset}{dZ^4} - \left(GC_s + K_p\right)\frac{d^2\emptyset}{dZ^2} + K_w\emptyset - \rho I_p \frac{d^2\emptyset}{dt^2} = 0 \tag{1}$$

For free torsional vibrations, the angle of twist $\emptyset(z,t)$ can be expressed in the form,

$$\emptyset(z,t) = x(z)e^{i\omega t} \tag{2}$$

In which $x(z)$ is the modal shape function corresponding to each beam torsional natural frequency ω.

The expression for $x(z)$ which satisfies Eq. (1) can be written as

$$x(z) = A\cos\beta L + B\sin\beta L + C\cosh\alpha L + D\ \sinh\alpha L \tag{3}$$

In which αL *and* βL are the positive, real quantities given by

$$\beta L, \alpha L = \sqrt{\frac{\mp(K^2 + {\gamma_P}^2) + \sqrt{(K^2 + {\gamma_P}^2)^2 + 4(\lambda^2 - {\gamma_W}^2)}}{2}} \quad (4)$$

$$K^2 = \left(\frac{G C_S L^2}{E C_W}\right);\ \gamma_P = \sqrt{\frac{K_P\ L^2}{E C_W}};\ \gamma_W = \sqrt{\frac{K_W\ L^4}{E C_W}};\ \lambda^4 = \left(\frac{\rho I_P \omega^2 L^4}{E C_W}\right) \quad (5)$$

From Eq. (4) we have the following relation between αL *and* βL

$$(\beta L)^2 = (\alpha L)^2 + K^2 - {\gamma_P}^2 \quad (6)$$

Knowing α *and* β, the frequency parameter λ can be evaluated using the relation

$$\lambda^2 = (\alpha L)(\beta L) + {\gamma_W}^2 \quad (7)$$

The four arbitrary constants A, B, C and D in Eq. (3) can be determined from the boundary conditions of the beam. For any single-span beam, there will be two boundary conditions at each end and these four conditions then determine the corresponding frequency and mode shape expressions.

DYNAMIC STIFFNESS MATRIX

In order to proceed further, we must first introduce the following nomenclature. The flange bending moment and the total twisting moment are given by M(z) andT(z). The variation of angle of twist $\emptyset$ with respect to z is denoted by $\theta(z)$. Considering anti-clockwise rotations and moments to be negative, we have

$$\theta(z) = d\emptyset/dz, hM(z) = -E C_w\, d^2\emptyset/dz^3 \quad (8)$$

$$T(Z) = -E C_W \frac{d^3\varphi}{dz^3} + G C_S \frac{d\varphi}{dZ} \quad (9)$$

Where EC_w is termed as the warping rigidity of the section,

$$\mathrm{E}C_w = \frac{I_f h^2}{2} \quad (10)$$

Consider a uniform thin-walled I-beam element of length L as shown in the Figure 3. By combining the Eq. (3) and Eq. (8), the end displacements, $\emptyset(0)$and $\theta(0)$ and end forces $hM(0)$ and $T(0)$, of the beam at $z = 0$, can be expressed as

$$\begin{Bmatrix} \emptyset(0) \\ \theta(0) \\ hM(0) \\ T(0) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \alpha & 0 & \beta \\ EC_W\alpha^2 & 0 & EC_W\beta^2 & 0 \\ 0 & EC_W\alpha\beta^2 & 0 & -EC_W\alpha^2\beta \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} \tag{11}$$

Eq. (3.28) can be abbreviated as follows

$$\delta(0) = S(0)U \tag{12}$$

In a similar manner, the end displacements$\emptyset(L)$,$\theta(L)$ and end forces$hM(L)$, $T(L)$of the beam at $z = L$, can be expressed as

$$\delta(L) = S(L)U \tag{13}$$

Where

$$S(L) = \begin{Bmatrix} \emptyset(L) \\ \theta(L) \\ hM(L) \\ T(L) \end{Bmatrix} \tag{14}$$

$$\{U\}^T = \{A \quad B \quad C \quad D\} \tag{15}$$

$$[S(L)] = \begin{bmatrix} c & s & C & S \\ -\alpha s & \alpha c & \beta S & \beta C \\ EC_W\alpha^2 c & EC_W\alpha^2 s & -EC_W\beta^2 C & -EC_W\beta^2 S \\ -EC_W\alpha\beta^2 s & EC_W\alpha\beta^2 c & -EC_W\alpha^2\beta S & -EC_W\alpha^2\beta C \end{bmatrix} \tag{16}$$

In which

$$c = \cos\beta L, s = \sin\beta L, C = \cosh\alpha L, S = \sinh\alpha L \tag{17}$$

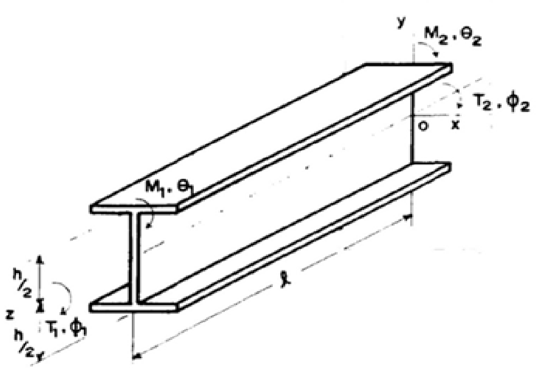


Figure 4: Differential Element of Thin Wall I Beam

The equation relating the end forces and displacements can be written as

$$\begin{Bmatrix} T(0) \\ hM(0) \\ T(L) \\ hM(L) \end{Bmatrix} = \begin{bmatrix} 0 & EC_W\alpha\beta^2 & 0 & -EC_W\alpha^2\beta \\ EC_W\alpha^2 & 0 & EC_W\beta^2 & 0 \\ -EC_W\alpha\beta^2 s & EC_W\alpha\beta^2 c & -EC_W\alpha^2\beta S & -EC_W\alpha^2\beta C \\ EC_W\alpha^2 c & EC_W\alpha^2 s & -EC_W\beta^2 C & -EC_W\beta^2 S \end{bmatrix} * [U] \begin{Bmatrix} \emptyset(0) \\ \theta(0) \\ \emptyset(L) \\ \theta(L) \end{Bmatrix} \quad (18)$$

By eliminating the integration constant vector U and designating the left end element as I and the right end as j, the final equation relating the end forces and displacements can be written as

$$\begin{Bmatrix} T_i/EC_W \\ hM_j/EC_W \\ T_j/EC_W \\ hM_j/EC_W \end{Bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22} & J_{23} & J_{24} \\ J_{31} & J_{32} & J_{33} & J_{34} \\ J_{41} & J_{42} & J_{43} & J_{44} \end{bmatrix} \begin{Bmatrix} \varphi_i \\ \theta_i \\ \varphi_j \\ \theta_j \end{Bmatrix} \quad (19)$$

Eq. (19) is symbolically written as

$$\{F\} = [J]\{U\} \quad (20)$$

In the Eq. (20) the matrix $[J]$is the 'exact' element dynamic stiffness matrix (DSM), which is also a square matrix. The elements of $[J]$are

$$J_{11} = H(\alpha^2 + \beta^2)(\alpha Sc + \beta Cs)$$

$$J_{12} = -H[(\alpha^2 - \beta^2)(1 - Cc) + 2\alpha\beta Ss]$$

$$J_{13} = -H(\alpha^2 + \beta^2)(\alpha s + \beta S)$$

$$J_{14} = -H(\alpha^2 + \beta^2)(C - c)$$

$$J_{22} = -(H/\alpha\beta)(\alpha^2 + \beta^2)(\alpha Sc - \beta Cs)$$

$$J_{24} = (H/\alpha\beta)(\alpha^2 + \beta^2)(\alpha S - \beta s)$$

$$J_{23} = -J_{14};\ J_{33} = J_{11};\ J_{34} = -J_{12};\ J_{44} = J_{22}$$

$$H = \alpha\beta/[2\alpha\beta(1 - Cc + (\beta^2 - \alpha^2)Ss] \quad (21)$$

Using the element dynamic stiffness matrix defined by Eq. (20), one can easily set up the general equilibrium equations for multi-span thin-walled beams, adopting the usual finite element assembly methods. Introducing the boundary conditions, the final set of equations can be solved for eigenvalues by setting up the determinant of their matrix to zero

METHOD OF SOLUTION

Denoting the modified dynamic stiffness matrix as [J], we state that

$$det|J| = 0 \quad (22)$$

The above equation yields the frequency equation of continuous thin-walled beams in torsion resting on Winkler-Pasternak type foundation. It can be noted that above equation is highly transcendental, the roots of equation can, therefore, be obtained by applying the bisection method using MATLAB code on a high-speed digital computer.

A new MATLAB code was developed based on bisection method and was published in MATLAB Central official online library [29] which was cited and referred by few researchers.

- These are some of the key highlights of the new MATLAB code
- Primarily, it solves almost any given linear, non-linear & highly-transcendental equations.
- Additional key highlight of this code is, the equation whose roots are to be found, can be defined separately in an ".m" file, which facilitates to solve multi-variable (example$x^2 + y^2 + \sin x + \cos y = 0$) highly-transcendental equations of any size, where the existing MATLAB codes fail to compute.
- This code is made robust in such a manner that it can automatically save and write the detailed informationsuch as no. of iterations; the corresponding values of the variables and the computing time are automatically saved, into a (.txt) file format, in a systematic tabular form.
- This code has been tested on MATLAB 7.14 (R2012a) [30] for all possible types of equations and proved to be accurate.

Exact values of the frequency parameter λ for various boundary conditions of thin-walled open section beam are obtained and the results are presented graphical form in this paper for varying values of warping, Winkler foundation and Pasternak foundation parameters.

RESULTS AND DISCUSSIONS

The approach developed in this paper can be applied to the calculation of natural torsional frequencies and mode shapes of multi-span doubly symmetric thin-walled beams of open section such as beams of I-section. Beams with non-uniform cross-sections also can be handled very easily as the present approach is almost similar to the finite element method of analysis but with exact displacement shape functions. All classical and non-classical (elastic restraints) boundary conditions can be incorporated in the present model without any difficulty.

In the following, the end conditions for this problem can be written as

$\left.\begin{matrix}\emptyset = 0\\ M = 0\end{matrix}\right\} pinned\ end$	$\left.\begin{matrix}\emptyset = 0\\ \theta = 0\end{matrix}\right\} fixed\ end$
$\left.\begin{matrix}T = 0\\ M = 0\end{matrix}\right\} free\ end$	$\left.\begin{matrix}\theta = 0\\ T = 0\end{matrix}\right\} guided\ end$

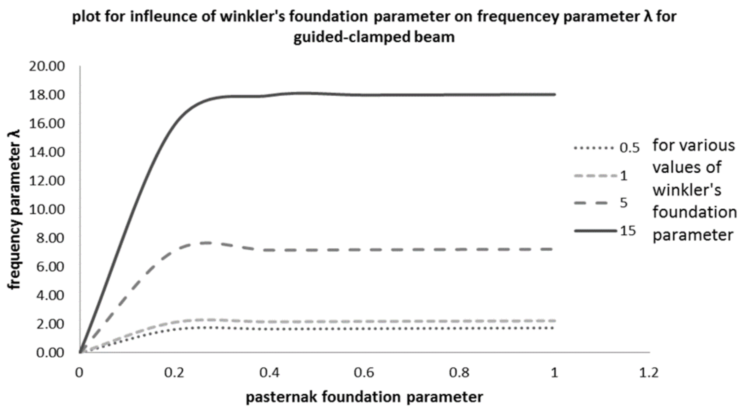
Considering a one element solution and applying these boundary conditions to Eq. (19) gives,

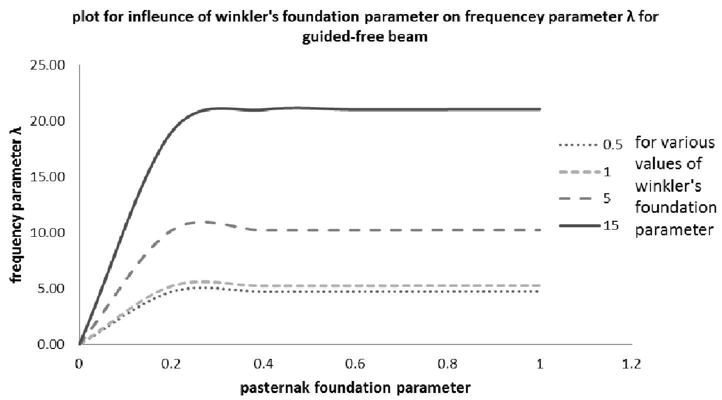
Table 1: Comparison of Obtained Frequency Equations with Ref. [32]

S No.	Boundary Condition	Frequency Equations (Present Work)	Frequency Equations (Ref. [32])
1	Guided-guided	$\sin(\beta L) * \sinh(\alpha L) = 0$	$\sin(y_n) = 0$
2	Guided-free	$\alpha^3 \tan(\beta L) + \beta^3 \tanh(\alpha L) = 0$	$\tan(y_n) + \tanh(y_n.) = 0$
3	Guided-pinned	$\frac{2\alpha^2\beta^2}{(\alpha^4 + \beta^4)} \cosh(\alpha L) \cos(\beta L) + 1 = 0$	$\cosh(y_n) \cos(y_n) = -1$
4	Guided-clamped	$\beta \tan(\beta L) + \alpha \tanh(\alpha L) = 0$	$\tan(y_n) + \tanh(y_n) = 0$

The transcendental frequency equations (Eq. (4a), Eq. (4b), Eq. (4e) and Eq. (4f)) of Ref. [32], for generally restrained beams are observed to be same as equations obtained in the present work (see Table 1) of the present paper, but for only difference that the roots α and β are considered to be equal and have same sign and defined as (y_n).

Furthermore, the equations for all the BC's are also solved for values of warping parameter $0 \leq K \leq 20$ and for various values of Winkler foundation parameter $0 \leq \gamma_w \leq 15$ and values of Pasternak foundation parameter $0 \leq \gamma_p \leq 1$ and are presented in graphical form for the first mode.





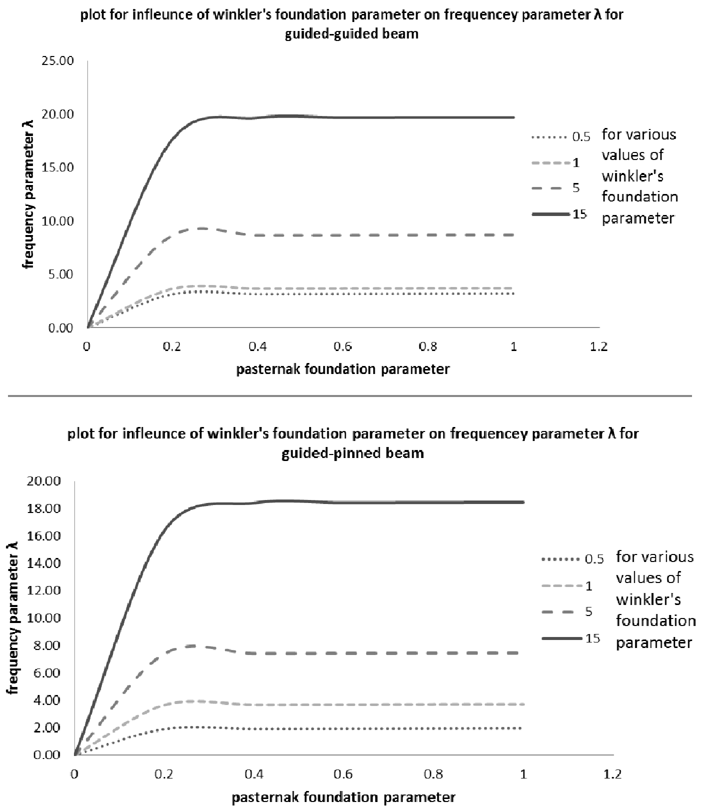


Figure 5: Plot for Influence of Winkler Foundation Parameter on Frequency Parameter for Values of Pasternak Foundation Parameter for Various BC's

The influences of the foundation parametersγ_w *and* γ_p, on the stability parameter for guided-end conditions is shown in the Figure 5, The figures indicate that the stability parameter increases as the overall stiffness of the beam-foundation system increases. The overall stiffness of the beam-foundation system is an integrated resultant of the support stiffness, the foundation stiffness and the flexural rigidity of the beam. It is known that the flexural rigidity of the beam increases as λ increases. It is obvious that the frequency parameter increases as the overall stiffness of the beam foundation system increases.

The plots clearly show that while the Winkler foundation independently increases the frequency for any mode of vibration for constant values of warping and the Pasternak foundation parameters. Interestingly we can clearly observethat the effect of Pasternak foundation Parameter is to decrease the natural torsional frequency significantly for any mode of vibration and for constant values of warping and Winkler foundation parameter.

The influence warping parameter Kon the stability parameter for guided-end conditions is shown in the Figure 6 for all guided-end conditions. A close look at the results presented in Figure 6 clearly reveals that the effect of an increase in warping parameter K is to drastically decrease the fundamental frequencyλ. Furthermore, can be expected, the effect of elastic foundation is found to increase the frequency of vibration especially for the first few modes. However, this influence is seen to be quite negligible on the modes higher than the third.

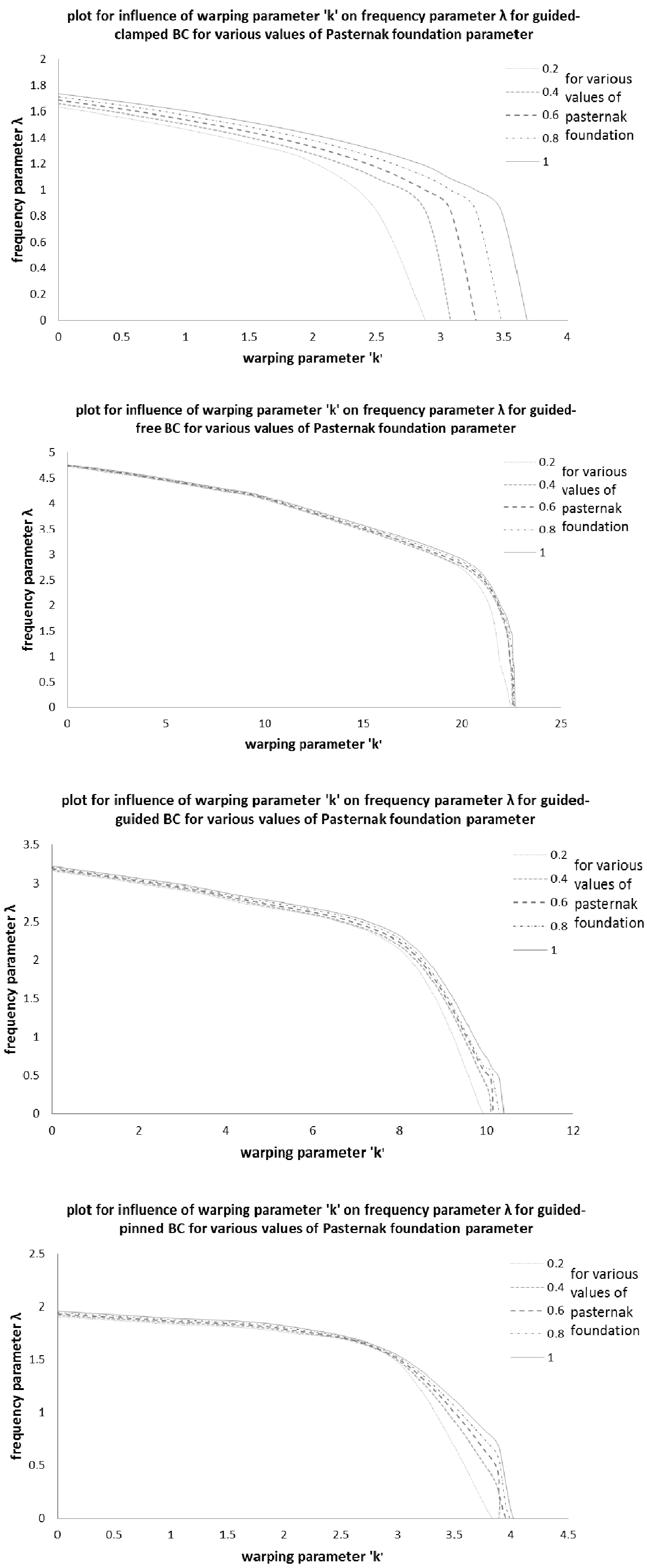


Figure 6: Plot for Influence of Warping Parameter on Frequency Parameter for Values of Pasternak Foundation Parameter and Winkler Foundation Parameter for Guided-End BC's

CONCLUDINGS REMARKS

In this paper, a dynamic stiffness matrix (DSM) approach has been developed for computing the natural torsion frequencies of long, doubly-symmetric thin-walled beams of open section resting on continuous Winkler-Pasternak type foundation. The approach presented in this paper is quite general and can be applied for treating beams with non-uniform cross-sections and also non-classical boundary conditions. Numerical results for natural frequencies for various values of warping and Winkler and Pasternak-foundation parameters are obtained and presented as graphical form showing their parametric influence clearly. From the results obtained following conclusions are drawn.

- An attempt was made to validate the present formulation of the problem for guided-end conditions. There is very good agreement between the results, the general dynamic stiffness matrix defined by Eq. (33) and Eq. (47) of Yung-Hsiang Chen [15], for Euler-Bernoulli beam is observed to be same as Eq. (19) and Eq. (20) in this paper, but for only difference that the axial force was not included in the present paper.

- The influences of the foundation parameters γ_w , γ_p and warping parameter K, on the stability parameter λ for various supporting conditions are shown in the Figures (5 and 6). The second foundation parameter γ_p, tends to increase the fundamental frequency for the same Winkler constant γ_w. The effect of γ_p, may be interpreted in the following way: A simply guided-pinned beam, which is the weakest as far as stability ($\lambda = 1.90$) is concerned, acquires the stability of a beam which is guided at one end and free at other end ($\lambda = 5.22$), by increasing the shear parameter of the foundation especially for the first mode. However, this influence is seen to be quite negligible on the modes higher than the first. Also, it is found that the effect of an increase in warping parameter K is to drastically decrease the stability parameter λ.

It can be finally concluded that for an appropriately designing the thin-walled beams of open cross sections resting on continuous elastic foundation, it is very much necessary to model the foundation appropriately considering the Winkler and Pasternak foundation stiffness values as their combined influence on the natural torsional frequency is quite significant and hence cannot be ignored.

NOMENCLATURE

Table 2

T_s	St. Venant's torsion
T_w	warping torsion
T_n	Total Non-Uniform Torsion
K	Modulus Of Subgrade Reaction
P	Pressure
G_p	Shear Modulus
$\emptyset$	Angle Of Twist
G	Modulus Of Rigidity
C_s	Shear Constant
M	Twisting Moment In Each Flange
h	Distance Between The Center Lines Of The Flanges
I_f	Moment Of Inertia Of Flange About Its Strong Axis
u	Lateral Displacement Of The Flange Centerline
C_w	Warping Constant

E	Young's Modulus
ρ	Mass Density Of The Material Of The Beam
I_p	Polar Moment Of Inertia
K_w	Winkler Foundation Stiffness
K_p	Pasternak Layer Stiffness
Z	Distance Along The Length Of The Beam
ω	Torsional Natural Frequency
K	Non-Dimensional Warping Parameter
γ_P	Non-Dimensional Pasternak Foundation Parameter
γ_W	Non-Dimensional Winkler Foundation Parameter
λ	Non-dimensional frequency parameter
$\theta(z)$	Variation Of Angle Of Twist ∅

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